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On Persistence of Spatial Analyticity for the Higher-Order Even Dispersion Cubic Nonlinear Schrodinger Equation on the Circle

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Abstract

We consider the initial value problem on the circle for a cubic nonlinear schrodinger equation governed by a higher-order even dispersion operator D^{α} with an even integer exponent $\alpha>2$. Our main result proves that the uniform radius of spatial analyticity $\sigma(t)$ survives over time, satisfying a lower bound decay rate of $c|t|^{-1}$ for a positive constant c. To overcome the derivative losses and track the evolution of the analytic radius, we construct a higher-order almost conservation law in Gevrey spaces. The analytical framework incorporates Duhamel's formula, Strichartz estimates for the linear wave, trilinear estimates, Plancherel's identity, Holder's inequality, and Sobolev embeddings.

Keywords: *Fractional Nonlinear Schrodinger Equation, Gevrey Spaces, Lower Bound for the Radius of Spatial Analyticity.*

1991 Mathematics Subject Classification. 35Q55, 35A01, 35A20, 42B35, 35B65, 35G25, 35B10

1. Introduction

Consider the Initial Value Problem (IVP) for the cubic Nonlinear Schrodinger (NLS) equation with higher-order even dispersion on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

$$(\alpha\text{-NLS}) \qquad \begin{cases} i\,\partial_t u + D^{\alpha}u = |u|^2 u, \\ u(x,0) = u_0(x), \end{cases}$$

where $\alpha > 0$ and $u := u(x, t)$ is a complex-valued function. The operator $D^{\alpha} = (-\partial_x^2)^{\alpha/2}$ is defined via the Fourier transform by $F(D^{\alpha} f)(\xi) = |\xi|^{\alpha} \hat{f}(\xi)$.

The mass and energy functionals, which are conserved by the flow of (α-NLS), are de-fined as:

$$(1.1) \qquad M(t) = \int_{\mathbb{T}} |u|^2\, dx,$$

$$(1.2) \qquad E(t) = \int_{\mathbb{T}} \left|D^{\alpha/2}u\right|^2 dx + \frac{1}{2}\int_{\mathbb{T}} |u|^4\, dx.$$

The fractional Schrödinger equations were first introduced in the theory of the fractional quantum mechanics, where the Feynmann path integral approach is generalized to α-stable Levy process [1]. Different choices of α represent different physical contexts. For example, for $\alpha = 2$, the equation is the cubic NLS equation which appears to be an important model in the study of nonlinear optics, fluids and plasma physics [2] . For $\alpha = 3$, equation (α-NLS) becomes the third-order NLS (3NLS) equation, which is a mathematical model for nonlinear pulse propagation phenomena in various fields of physics, especially in nonlinear optics [3][4] . For $\alpha = 4$, equation (α-NLS) corresponds to the cubic fourth-order NLS (4NLS) and has applications in the study of solitons in magnetic materials [5][6]. We also note that fNLS for $0 < \alpha < 2$ arises in various physical settings [7][8][9][10].

The cubic fNLS is well-posed in H^s for $s > \frac{1-\alpha}{2}$ and ill-posed in H^s for $s < \frac{1-\alpha}{2}$. For instance, for the cubic NLS (i.e. (α-NLS) with $\alpha = 2$) on $\mu = \mathbb{R}$ or $\mathbb{T}$, Christ-Colliander-Tao [11] and Kishimoto [12] proved the ill-posedness of (α-NLS) in the sense of norm inflation in $H^s(\mathcal{M})$ for $s \le -\frac{1}{2}$; see also [13][14]. Indeed, in [15], Choffrut-Pocovnicu proved norm inflation of (α-NLS) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ above s_c: when $\alpha > 2$, norm inflation of (α-NLS) holds in $H^s(\mathcal{M})$ for $s < \frac{1-2\alpha}{6}$, which is above $s_c = \frac{1-\alpha}{2}$ given that $\alpha > 2$.

There are also results for other types of ill-posedness of (α-NLS) above the scaling critical regularity. For example, for the cubic NLS (i.e., $\alpha = 2$) on $\mathbb{T}$, Christ-Colliander-Tao [16] and Molinet [17] showed failure of continuity of the solution map in $H^s(\mathbb{T})$ for $s < 0$. Moreover, Guo-Oh [18] and Oh-Wang [19] proved the non-existence of solutions of the cubic NLS ($\alpha = 2$) and 4NLS ($\alpha = 4$) on $\mathbb{T}$ if the initial data lies in $H^s(\mathbb{T})\backslash L^2(\mathbb{T})$, where $s \in \left(-\frac{1}{8}, 0\right)$ and $s \in \left(-\frac{9}{20}, 0\right)$, respectively.

In [20], Brun *et al.* studied the cubic fractional Schrödinger equation (α-NLS), and established local well-posedness in $H^s(\mathbb{R})$ for $s \geq \frac{2-\alpha}{4}$ with $\alpha > 2$ and in $H^s(\mathbb{T})$ for $s \geq \frac{2-\alpha}{6}$ was established. Furthermore, they extended these local solutions globally in time for $s \geq \frac{2-\alpha}{4}$ on the line and for $s \geq \frac{2-\alpha}{6}$ on the torus.

The main concern of this paper for the cubic fNLS (α-NLS) is to study the spatial analyticity of the solution $u(x,t)$. In particular, we consider real analytic initial data $u_0(x)$ with a uniform radius of analyticity $\sigma_0 > 0$, so that there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy \in \mathbb{C}: |y| < \sigma_0\}.$$

The question is then, is the solution $u(x,t)$ of (α-NLS) analytic in $S_{\sigma(t)}$ for all times t? The novelty of this work lies in the extension of the persistence of spatial analyticity to higher-order even dispersion ($\alpha \geq 4$) on the torus. While the case $\alpha = 2$ is well-understood [21], the higher-order dispersion introduces significant technical difficulties. Specifically, the derivative losses in the error terms of the almost conservation law are more severe, requiring a more refined multilinear analysis in Gevrey-Bourgain spaces. We overcome these difficulties by establishing a higher-order almost conservation law and a rigorous global extension argument that ensures a uniform time step.

The study of the global analyticity of nonlinear partial differential equations (PDEs) was first initiated by Kato and Masuda [22]. Since then, many scholars have devoted much effort to obtaining global analytic solutions for a number of nonlinear PDEs (see, e.g., [23][24][25][26] and the references therein). In particular, the global analyticity of (α-NLS) was investigated on the real line in [27] for $\alpha = 2$, in [28] for $\alpha = 4$, and in [29][30] for $\alpha = 3$. Moreover, on the circle, Tesfahun [31] studied the persistence of spatial analyticity for the case $\alpha = 2$.

The Gevrey space $G^{\sigma,s} := G^{\sigma,s}(\mathbb{T})$ for $\sigma > 0$ and $s \in \mathbb{R}$ is defined as follows

$$\| f \|^2_{G^{\sigma,s}} = \sum_{k\in\mathbb{Z}} e^{2\sigma|k|} \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2 \quad (\sigma \geq 0, s \in \mathbb{R})$$

where $\langle k\rangle = (1 + |k|^2)^{1/2}$ and $\hat{f}$ denotes the spatial Fourier transform given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ikx} f(x)dx \quad (k \in \mathbb{Z}),$$

So the inversion formula reads

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{ikx} \hat{f}(k)$$

and we have Plancherel's identity

$$\int_{\mathbb{T}} f(x)g(x)dx = \sum_{k\in\mathbb{Z}} \hat{f}(k)\hat{g}(k).$$

For $\sigma = 0$ this space reduces to the Sobolev space $H^s := H^s(\mathbb{T})$, with norm

$$\| f \|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left|\hat{f}(k)\right|^2,$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{T})$ has a radius of analyticity of at least σ at each point $x \in \mathbb{T}$ as the following theorem states whose proof is given for $s = 0$ in [32] applies also for $s \in \mathbb{R}$ with some obvious modifications.

Paley-Wiener Theorem: Let $\sigma > 0$ and $s \in \mathbb{R}$. A 2π-periodic function $f(x)$ belongs to $G^{\sigma,s}(\mathbb{T})$ if and only if it is the restriction to the real line of a function $F(x + iy)$ which is 2π-periodic in x, holomorphic in the strip

$$S_\sigma = \{x + iy: |y| < \sigma\},$$

and satisfies the bound

$$\sup_{|y|<\sigma} \| F(x + iy) \|_{H^s(\mathbb{T})} < \infty.$$

We note the following embedding property of the Gevrey spaces:

(1.3) $\quad G^{\sigma,s} \hookrightarrow G^{\sigma',s'} \quad$ for all $0 \leq \sigma' < \sigma$ and $s, s' \in \mathbb{R}$.

In particular, setting $\sigma' = 0$, we have the embedding $G^{\sigma,s} \hookrightarrow G^{s'}$ for all $0 < \sigma$ and $s, s' \in \mathbb{R}$. As a consequence of this embedding property and the existing well-posedness theory in $H^s(\mathbb{R})$, the IVP (α-NLS) with initial data $u_0 \in G^{\sigma_0,s}$ (for all $\sigma_0 > 0$ and $s \in \mathbb{R}$) has a unique, global-in-time solution.

Theorem 1.1 (Lower bound on the radius of spatial analyticity). Let $u_0 \in G^{\sigma_0,\frac{\alpha}{2}}$ for $\sigma_0 > 0$ and any even integer $\alpha \geq 4$. Then, the global solution u of (α-NLS) satisfies

$$u(t) \in G^{\sigma,\frac{\alpha}{2}} \quad \text{for all } t \in \mathbb{R},$$

where the radius of analyticity $\sigma := \sigma(t)$ satisfies the asymptotic lower bound

$$(1.4) \qquad \sigma \geq c|t|^{-1} \quad \text{as } |t| \to \infty,$$

for a constant $c > 0$ depending on $\| u_0 \|_{G^{\sigma_0,\frac{\alpha}{2}}}$.

This paper is organized as follows. In Section 2, we introduce our notation, the relevant function spaces, and various estimates. Morovere, we present the local well-posedness result. Finally, in Section 3, we prove an almost coservation law and the main result stated in Theorem 1.1

2. Preliminaries

In this section, we discuss notations, function spaces, and some relevant preliminary estimates.

2.1. Notations and Function Spaces

For $a, b > 0$, we write $a \lesssim b$ (resp. $a \sim b$) to mean there is a positive constant C such that $a \leq Cb$ (resp. $C^{-1}b \leq a \leq Cb$). Moreover, we use $a\pm$ to mean $a \pm \epsilon$ for some constant $\epsilon > 0$.

For $s, b \in \mathbb{R}$, the Bourgain space is denoted by $X^{s,b} = X^{s,b}(\mathbb{R} \times \mathbb{T})$, which is defined by the norm

$$\| u \|_{X^{s,b}} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau + |k|^{\alpha} \rangle^{2b} |\tilde{u}(\tau,k)|^2 d\tau\right)^{1/2},$$

Where $\tilde{u}(\tau,k)$ is the space-time Fourier transforms given by

$$\tilde{u}(\tau,k) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-i(kx+\tau t)}\, u(t,x)dxdt \quad (k \in \mathbb{Z}, \tau \in \mathbb{R}),$$

whose inversion reads

$$(2.1) \quad u(t,x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(kx+\tau t)}\, \tilde{u}(\tau,k) d\tau.$$

Moreover, for any time $\delta > 0$, we define the following restricted space:

$$\| u \|_{X^{s,b}_{\delta}} := \inf\{\| v \|_{X^{s,b}} : v = u \text{ on } \mathbb{T} \times (0,\delta)\}.$$

For $\sigma > 0$ and $s, b \in \mathbb{R}$, we define the Gevrey-Bourgain space $X^{\sigma,s,b} := X^{\sigma,s,b}(\mathbb{R} \times \mathbb{T})$ as the completion of the Schwartz space with respect to the norm

$$(2.2) \quad \| u \|_{X^{\sigma,s,b}} = \| e^{\sigma|\xi|} \langle \xi \rangle^{s} \langle \tau + |\xi|^{\alpha} \rangle^{b} |\tilde{u}(\xi,\tau)| \|_{L^2_{\tau,\xi}}.$$

Additionally, for $\delta > 0$, we denote the Gevrey-Bourgain space restricted in time by $X^{\sigma,s,b}_{\delta}$, with the norm defined as

$$\| u \|_{X^{\sigma,s,b}_{\delta}} = \inf\{\| v \|_{X^{\sigma,s,b}} : v = u \text{ on } \mathbb{T} \times (0,\delta)\}.$$

When $\sigma = 0$, we recover the standard Bourgain spaces: $X^{0,s,b} \equiv X^{s,b}$ and $X^{0,s,b}_{\delta} \equiv X^{s,b}_{\delta}$.

2.2. Preliminary Estimates

Consider the IVP for the linear Schrödinger equation with higher- order even dispersion:

$$(2.3) \qquad \begin{cases} i\,\partial_t u + D^{\alpha} u = F, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases}$$

for given $u_0 \in G^{\sigma,s}(\mathbb{T})$ and $F \in X^{\sigma,s,b-1}_{\delta}$ with $\sigma \geq 0, s \in \mathbb{R}, 1/2 < b \leq 1$ and $0 < \delta \leq 1$. The solution of (2.3) is given by the Duhamel's formula:

$$u(t) = S(t)u_0 - i\int_0^t S\,(t - t')F(t')dt',$$

and satisfies the following energy estimates:

$$(2.4) \qquad \| S(t)u_0 \|_{X^{\sigma,s,b}_{\delta}} \lesssim \| u_0 \|_{G^{\sigma,s}},$$

$$(2.5) \qquad \| \int_0^t S\,(t - t')F(t')dt' \|_{X^{\sigma,s,b}_{\delta}} \lesssim \| F \|_{X^{\sigma,s,b-1}_{\delta}}.$$

where $S(t)$ is unitary group given by $\widehat{S(t)u_0} = e^{-it|\xi|^{\alpha}}\widehat{u_0}$.

We also recall the following L^4-Strichartz estimate from.

$$(2.6) \qquad \| u \|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \leq C \| u \|_{X^{0,b}}$$

for all $u \in X^{0,b}$ with $b \geq \frac{\alpha+1}{4\alpha}$ and $\alpha > 2$.

For $\sigma \geq 0, s \in \mathbb{R}$ and $-1/2 < b < b' < 1/2$, we have

$$(2.7) \qquad \| u \|_{X^{\sigma,s,b}_{\delta}} \leq C\delta^{b'-b} \| u \|_{X^{\sigma,s,b'}_{\delta}},$$

$$(2.8) \qquad \| \chi_{(0,\delta)} u \|_{X^{\sigma,s,b}} \leq C \| u \|_{X^{\sigma,s,b}_{\delta}},$$

where $\chi_{(0,\delta)}$ is the characteristic function of $(0,\delta)$, and C is a positive constant depending only on b and b'. The proof for the case $\sigma = 0$ can be found in [33], while for $\sigma > 0$, the $X^{\sigma,s,b}$-estimates follow easily by the substitution $u \to e^{\sigma|D|}u$ from the properties of standard $X^{s,b}$-spaces (and their restrictions).

The following trilinear estimates will be key ingredients in the proof of local existence and almost conservation law.

Lemma 1. Let U_j denotes u_j or $\bar{u}_j$. For $\sigma \geq 0$ and $b > 1/2$

we have the estimates

(2.9) $$\left\|\prod_{j=1}^{3} U_j\right\|_{X^{0,-b}} \leq C \prod_{j=1}^{3} \| u_j \|_{X^{0,b}},$$

(2.10) $$\left\|\prod_{j=1}^{3} U_j\right\|_{L^2_{x,t}} \leq C \prod_{j=1}^{2} \| u_j \|_{X^{1,b}_{\delta}} \| u_3 \|_{X^{0,b}_{\delta}},$$

(2.11) $$\left\|\prod_{j=1}^{3} U_j\right\|_{X^{\sigma,1,0}} \leq C \prod_{j=1}^{3} \| u_j \|_{X^{\sigma,1,b}}.$$

Proof. By duality, the estimate (2.9) is equivalent to proving that for any function $u_4 \in X^{0,b}(\mathbb{T} \times \mathbb{R})$,

$$\left|\iint_{\mathbb{T}} \int_{\mathbb{R}} U_1\, U_2 U_3 \bar{u}_4\, dt\, dx\right| \leq C \prod_{j=1}^{4} \| u_j \|_{X^{0,b}}.$$

By Holder's inequality and (2.6), we have

$$\left|\iint_{\mathbb{T}} \int_{\mathbb{R}} U_1\, U_2 U_3 \bar{u}_4\, dt\, dx\right| \leq \prod_{j=1}^{4} \| u_j \|_{L^4_{t,x}} \leq$$

$$\prod_{j=1}^{4} \| u_j \|_{X^{0,b}} \leq C \prod_{j=1}^{3} \| u_j \|_{X^{0,b}}.$$

Similarly, for $u_4 \in L^2_{t,x}(\mathbb{T} \times \mathbb{R})$, (2.11) reduces to

$$\left|\iint_{\mathbb{T}} \int_{\mathbb{R}} U_1\, U_2 U_3 \bar{u}_4\, dt\, dx\right|$$

$$\leq C \| u_1 \|_{X^{1,b}} \| u_2 \|_{X^{1,b}} \| u_3 \|_{X^{0,b}} \| u_4 \|_{L^2_{t,x}(\mathbb{T}\times\mathbb{R})}.$$

We recall the following Sobolev embedding:

$$W^{s,p}(\mathbb{T}) \hookrightarrow L^q(\mathbb{T}) \quad \text{whenever} \quad s \geq \frac{1}{p} - \frac{1}{q}.$$

To reach $q = \infty$, we set $\frac{1}{q} = 0$, which yields the threshold $s > \frac{1}{p}$. For $p = 4$, this confirms that $s > \frac{1}{4}$ is required. Applying the standard 1D Sobolev embedding directly yields

$$\| u \|_{L^\infty_x} \lesssim \| u \|_{W^{s,4}_x} = \| \langle D_x \rangle^s u \|_{L^4_x}.$$

Since $s = 1 > \frac{1}{4}$, we obtain

(2.12) $$\| u \|_{L^\infty_x} \lesssim \| \langle D_x \rangle u \|_{L^4_x}.$$

By Holder's inequality, (2.12), and (2.6), we have

$$\left|\iint_{\mathbb{T}} \int_{\mathbb{R}} U_1\, U_2 U_3 \bar{u}_4\, dt\, dx\right|$$

$$\leq \| u_1 \|_{L^4_t L^\infty_x} \| u_2 \|_{L^4_t L^\infty_x} \| u_3 \|_{L^\infty_t L^2_x} \| u_4 \|_{L^2_{t,x}}$$
$$\leq \| \langle D_x \rangle u_1 \|_{L^4_{t,x}} \| \langle D_x \rangle u_2 \|_{L^4_{t,x}} \| u_3 \|_{L^\infty_t L^2_x} \| u_4 \|_{L^2_{t,x}}$$
$$\leq \| u_1 \|_{X^{1,b}} \| u_2 \|_{X^{1,b}} \| u_3 \|_{X^{0,b}} \| u_4 \|_{L^2_{t,x}}$$
$$\leq C \| u_1 \|_{X^{1,b}} \| u_2 \|_{X^{1,b}} \| u_3 \|_{X^{0,b}}.$$

Finally, since $k = k_1 + k_2 + k_3$, the standard triangle inequality for absolute values implies

$$|k| \leq |k_1| + |k_2| + |k_3| \Longrightarrow |k| - \sum_{j=1}^{3} |k_j| \leq 0.$$

Since $\sigma > 0$, multiplying by σ and taking the exponential yields

(2.13) $$e^{\sigma\left(|k| - \sum_{j=1}^{3} |k_j|\right)} \leq 1.$$

Thus, taking the space-time Fourier transform and using (2.13), proving (2.11) for $u_4 \in X^{1,b}$ reduces to

$$\left\|\prod_{j=1}^{3} U_j\right\|_{X^{1,0}} \leq \prod_{j=1}^{4} \| u_j \|_{X^{1,b}}.$$

By Plancherel's identity, this further reduces to

$$\left\|\prod_{j=1}^{2} U_j \cdot \langle D_x \rangle U_3\right\|_{L^2_{t,x}(\mathbb{T}\times\mathbb{R})} \leq C \prod_{j=1}^{3} \| u_j \|_{X^{1,b}}.$$

This, in turn, follows from (2.10) because the estimate (2.10) holds for arbitrary functions. (2.9)

To prove the almost conservation law we need the following estimate whose prove can be found in.

Lemma 2. Let $\sigma > 0$ and $k_1, k_2, k_3 \in \mathbb{Z}$ such that $k = k_1 - k_2 + k_3$. Denote the minimum, medium and maximum of $\{|k_1|, |k_2|, |k_3|\}$ by k_{min}, k_{med} and k_{max}, then the following estimate holds:

$$1 - e^{-\sigma\left(\sum_{j=1}^{3} |k_j| - |k|\right)} \leq 12\sigma k_{med}.$$

2.3. Local Well-Posedness Result

Following the arguments presented in, applying a Picard iteration in the $X^{\sigma,s,b}_{\delta}$-space, and using the estimates (2.7), (2.4), (2.5) along with the trilinear estimate (2.11) on the iterates, one obtains the following local result.

Theorem 2.1. Let $\sigma > 0$, and let $\alpha > 2$ be an even integer. For any $u_0 \in G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})$, there exist a time $\delta = \delta\left(\| u_0 \|_{G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})}\right) > 0$ and a unique solution $u \in C\left([0, \delta]; G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})\right)$ to the IVP (α-NLS) on $\mathbb{T} \times [0, \delta]$.

Moreover, $\delta = c_0\left(1+\| u_0 \|_{G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})}\right)^{-a}$ for some constants $a, c_0 > 0$ depending only on α andσ. In particular, this argument gives the solution bound

(2.14) $$\| u \|_{X_\delta^{\sigma,\frac{\alpha}{2},b}} \leq C \| u_0 \|_{G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})},$$

where $C > 0$ depends only on b.

3. Almost Conservation Law and Proof of Theorem 1.1

In this section, we first prove an almost conservation law and then establish the main result of this paper stated in Theorem 1.1.

3.1. Almost Conservation Law

In this subsection, we derive an almost mass + energy conservation law for the function

$$v_\sigma(x,t) := e^{\sigma|D_x|}u(x,t),$$

where u is a local solution to (α-NLS), which in turn implies that $u(x,t) = e^{-\sigma|D_x|}v_\sigma(x,t)$. To do this, we define a modified mass + energy functional associated with the function v_σ by

(3.1) $$E_\sigma(t) := \frac{1}{2}\int_{\mathbb{T}} \left(|v_\sigma|^2 + \left|D^{\frac{\alpha}{2}}v_\sigma\right|^2 + \frac{1}{2}|v_\sigma|^4\right)dx.$$

For $\sigma = 0$, we have $e^{\sigma|\xi|} = 1$, and hence we obtain

$$E_0(t) = M(t) + E(t) = M(0) + E(0) = E_0(0), \quad \forall t.$$

This corresponds to the conserved mass + energy quantity.

However, this quantity fails to be conserved in time for $\sigma > 0$, and we will prove that it is almost conserved by establishing a growth estimate. To do this, we first apply the operator $\cosh(\sigma|D|)$ to the NLS equation in (α-NLS). Then, the function v_σ satisfies the equation

(3.2) $$i\,\partial_t v_\sigma + D^\alpha v_\sigma = |v_\sigma|^2 v_\sigma + K(v_\sigma),$$

where $K(v_\sigma)$ is given by

(3.3) $$K(v_\sigma) := -\left(|v|^2 v - e^{\sigma|D_x|}\left(\left|e^{-\sigma|D_x|}v\right|^2 e^{-\sigma|D_x|}v\right)\right).$$

Differentiating $E_\sigma(t)$ with respect to t, and applying (3.2), (3.3), and integration by parts[1], we obtain

$$\frac{d}{dt}E_\sigma(t)$$

$$= 2\mathrm{Re}\int_{\mathbb{T}}\left(\overline{v_\sigma}\,\partial_t v_\sigma + \overline{D^{\frac{\alpha}{2}}v_\sigma}D^{\frac{\alpha}{2}}\,\partial_t v_\sigma + |v_\sigma|^2\overline{v_\sigma}\,\partial_t v_\sigma\right)dx$$

(3.4) $$= 2\mathrm{Re}\int_{\mathbb{T}}\partial_t\, v_\sigma\left(\overline{v_\sigma + D^\alpha v_\sigma + |v_\sigma|^2 v_\sigma}\right)dx$$

$$= -2\mathrm{Re}\; i\int_{\mathbb{T}}\left(-D^\alpha v_\sigma + |v_\sigma|^2 v_\sigma + K(v_\sigma)\right)\left(\overline{v_\sigma + D^\alpha v_\sigma + |v_\sigma|^2 v_\sigma}\right)dx$$

$$= 2\mathrm{Im}\int_{\mathbb{T}} K(v_\sigma)\left(\overline{v_\sigma + D^\alpha v_\sigma + |v_\sigma|^2 v_\sigma}\right)dx.$$

Integrating (3.4) in time and using the self-adjointness of $D^{\frac{\alpha}{2}}$, we obtain

(3.5) $$E_\sigma(t) = E_\sigma(0) + I_\sigma(t),$$

where

(3.6) $$I_\sigma(t) := 2\mathrm{Im}\int_R\int_{\mathbb{T}}\chi_{(0,\delta)}(t)\left(\overline{v_\sigma}\cdot K(v_\sigma) + \overline{D^{\frac{\alpha}{2}}v_\sigma}\cdot D^{\frac{\alpha}{2}}K(v_\sigma) + |v_\sigma|^2\overline{v_\sigma}\cdot K(v_\sigma)\right)dxdt$$

for $K(v_\sigma)$ is as in (3.3)

Lemma 3. (Multilinear Estimates for $K(v_\sigma)$). For $\alpha \geq 4$ and $b > 1/2$, we have:

(3.7) $$\| K(v_\sigma) \|_{L^2_{t,x}} \lesssim \sigma \| v_\sigma \|^3_{X^{1,b}},$$

(3.8) $$\| D^{\alpha/2}K(v_\sigma) \|_{X^{0,-b}} \lesssim \sigma \| v_\sigma \|^3_{X^{\alpha/2,b}}.$$

Proof of Estimate (3.7) Taking the space-time Fourier transform of $K(v_\sigma)$, we have

$$\widetilde{K}(v)(\tau,k) = \sum_{k=k_1-k_2+k_3}\int_{\mathbb{R}^3}\left\{1 - e^{-\sigma\left(\sum_{j=1}^3|k_j| - |k|\right)}\right\}\widetilde{v_\sigma}(\tau_1,k_1)\overline{\widetilde{v_\sigma}(\tau_2,k_2)}\widetilde{v_\sigma}(\tau_3,k_3)\,d\mu$$

where $d\mu$ is a surface measure given by

$$d\mu = \delta\begin{pmatrix} k - k_1 + k_2 - k_3 \\ \tau - \tau_1 + \tau_2 - \tau_3 \end{pmatrix}d\tau_1 d\tau_2 d\tau_3.$$

Thus, we are summing over $k_1, k_2, k_3 \in \mathbb{Z}$ with the constraint $k_1 - k_2 + k_3 = k$ and integrating over k$\tau_1, \tau_2, \tau_3 \in \mathbb{R}$ such that $\tau_1 - \tau_2 + \tau_3 = \tau$.

Using Lemma 2, we obtain

$$\left|\widetilde{K}(v_\sigma)(\tau,k)\right| \leq 12\sigma \sum_{k=k_1-k_2+k_3} \int_{\mathbb{R}^3} k_{\text{med}} \, |\widetilde{v_\sigma}(\tau_1,k_1)||\widetilde{v_\sigma}(\tau_2,k_2)||\widetilde{v_\sigma}(\tau_3,k_3)| \, d\mu.$$

We can assume that v_σ and all its partial derivatives tend to zero as $|x| \longrightarrow \infty$ (see the detailed argument in [2]).

Let $\widetilde{w}_j(\tau_j,k_j) = |\widetilde{v_\sigma}(\tau_j,k_j)|$. By symmetry, we may assume $|k_1| \leq |k_2| \leq |k_3|$, and hence $k_{\text{med}} = |k_2|$. So, using symmetry, (2.10) and Sobolev embedding, we obtain

$$\begin{aligned}
&\| K(v_\sigma) \|_{L^2_{t,x}} \\
&= \|\widetilde{K}(v_\sigma)(\tau,k)\|_{L^2_\tau \ell^2_k} \\
&\lesssim \sigma \left\| \sum_{k=k_1-k_2+k_3} \int_{\mathbb{R}^3} |k_3| \, \widetilde{w}_1(\tau_1,k_1)\widetilde{w}_2(\tau_2,k_2)\widetilde{w}_3(\tau_3,k_3) \, d\mu \right\|_{L^2_\tau \ell^2_k} \\
&\lesssim \sigma \| w_1 \cdot w_2 \cdot |D_x| w_3 \|_{L^2_{t,x}} \\
&\lesssim \sigma \| w_1 \|_{X^{1,b}} \| w_2 \|_{X^{1,b}} \| |D_x| w_3 \|_{X^{0,b}} \\
&\lesssim \sigma \| v_\sigma \|^3_{X^{\frac{\alpha}{2},b}}. \qquad (3.7)
\end{aligned}$$

Proof of Estimate (3.8). Using a similar argument as in the proof of estimate (3.7), symmetry, (2.9), and Sobolev embedding, we obtain

$$\begin{aligned}
&\| D^{\frac{\alpha}{2}} K(v_\sigma) \|_{X^{0,-b}} \\
&= \left\| |k|^{\frac{\alpha}{2}} \langle \tau + k^2 \rangle^{-b} \widetilde{K}(v_\sigma)(\tau,k) \right\|_{L^2_\tau \ell^2_k} \\
&\lesssim \sigma \\
&\left\| \langle \tau + k^2 \rangle^{-b} \sum_{k=k_1-k_2+k_3} \int_{\mathbb{R}^3} |k_2| \, |k_3|^{\frac{\alpha}{2}} \widetilde{w}_1(\tau_1,k_1)\widetilde{w}_2(\tau_2,k_2)\widetilde{w}_3(\tau_3,k_3) \, d\mu \right\|_{L^2_\tau \ell^2_k} \\
&\sim \sigma \| w_1 \cdot |D_x| w_2 \cdot |D_x|^{\frac{\alpha}{2}} w_3 \|_{X^{0,-b}} \\
&\lesssim \sigma \| w_1 \|_{X^{1,b}} \| |D_x| w_2 \|_{X^{0,b}} \| |D_x|^{\frac{\alpha}{2}} w_3 \|_{X^{0,b}} \\
&\lesssim \sigma \| v_\sigma \|^3_{X^{\frac{\alpha}{2},b}}. \qquad (3.8)
\end{aligned}$$

After all, by applying Holder's inequality, using the estimates in Lemma 3 for the quantity $I_\sigma(t)$ defined in (3.6) and (2.10) and employing Sobolev embedding, we obtain

$$\begin{aligned}
I_\sigma(t) &\lesssim \| v_\sigma \|_{L^2_{t,x}([0,\delta]\times\mathbb{T})} \| \overline{K(v_\sigma)} \|_{L^2_{t,x}([0,\delta]\times\mathbb{T})} \\
&+ \| |v_\sigma|^2 v_\sigma \|_{L^2_{t,x}([0,\delta]\times\mathbb{T})} \| \overline{K(v_\sigma)} \|_{L^2_{t,x}([0,\delta]\times\mathbb{T})} \\
&+ \| D^{\frac{\alpha}{2}} v_\sigma \|_{X^{0,b}_\delta} \| \overline{D^{\frac{\alpha}{2}} K(v_\sigma)} \|_{X^{0,-b}_\delta} \\
&\lesssim \sigma \left(1 + \| v_\sigma \|^2_{X^{\frac{\alpha}{2},b}_\delta} \right) \| v_\sigma \|^4_{X^{\frac{\alpha}{2},b}_\delta}.
\end{aligned}$$

That is, for all $u \in X^{\sigma,\frac{\alpha}{2},b}_\delta$ with $b > \frac{1}{2}$, $0 < \delta \leq 1$, and $\sigma > 0$, we have

$$I_\sigma(t) \lesssim \sigma \left(1 + \| v_\sigma \|^2_{X^{\frac{\alpha}{2},b}_\delta} \right) \| v_\sigma \|^4_{X^{\frac{\alpha}{2},b}_\delta}. \qquad (3.9)$$

From (2.14), we have

(3.10)

$$\| v_\sigma \|_{X^{\frac{\alpha}{2},b}_\delta} = \| u \|_{X^{\sigma,\frac{\alpha}{2},b}_\delta} \leq C \| u_0 \|_{G^{\sigma,\frac{\alpha}{2}}(\mathbb{T})} \sim C \| v_\sigma(\cdot,0) \|_{H^{\frac{\alpha}{2}}(\mathbb{T})}.$$

On the other hand, following from the fact that $(a+b)^p \leq 2^p(a^p + b^p)$ for $a, b \geq 0$, and applying this to our case where $p = \frac{\alpha}{4}$ and $p = 2$ respectively:

$$\begin{aligned}
\| v_\sigma(\cdot,0) \|^2_{H^{\frac{\alpha}{2}}(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} \langle k \rangle^{2\frac{\alpha}{2}} |\widehat{v_\sigma}(k,0)|^2 \\
&\leq C \sum_{k\in\mathbb{Z}} 2^{\frac{\alpha+8}{4}} \left(1 + |k|^{2\left(\frac{\alpha}{2}\right)} \right) |\widehat{v_\sigma}(k,0)|^2 \\
&\leq C \int_{\mathbb{T}} \left(|v_\sigma(x,0)|^2 + \left| D^{\frac{\alpha}{2}} v_\sigma(x,0) \right|^2 \right) dx \\
&\leq C E_\sigma(0),
\end{aligned}$$

which in turn implies

$$\| v_\sigma(\cdot,0) \|_{H^{\frac{\alpha}{2}}(\mathbb{T})} \leq C \sqrt{E_\sigma(0)}. \qquad (3.11)$$

Hence, (3.11) combined with (3.10) gives

$$\| v_\sigma \|_{X^{\frac{\alpha}{2},b}_\delta} \leq C \sqrt{E_\sigma(0)}. \qquad (3.12)$$

Finally, in view of (3.12), (3.9), and (3.5), we get

$$E_\sigma(t) = E_\sigma(0) + C\sigma\big(1 + E_\sigma(0)\big) E^2_\sigma(0), \qquad (3.12)$$

which is an almost conservation law for the IVP (α-NLS).

3.2. Proof of Theorem 1.1

Let $u(\cdot,0) \in G^{\frac{\alpha}{2},\sigma_0}(\mathbb{T})$ for some $\sigma_0 > 0$ and for any even integer $\alpha > 2$. This implies $v_{\sigma_0}(\cdot,0) = e^{\sigma_0|D|} u(\cdot,0) \in H^{\frac{\alpha}{2}}$. Then, by the Sobelev embeddings[2], we have

$$\begin{aligned}
&E_{\sigma_0}(0) \\
&= \frac{1}{2} \| v_{\sigma_0}(\cdot,0) \|^2_{H^{\frac{\alpha}{2}}(\mathbb{T})} + \frac{1}{2} 4 \| v_{\sigma_0}(\cdot,0) \|^4_{L^4_x(\mathbb{T})}
\end{aligned}$$

$$\leq \frac{1}{2} \| v_{\sigma_0}(\cdot,0) \|^2_{H^{\frac{\alpha}{2}}(\mathbb{T})} + C \| v_{\sigma_0}(\cdot,0) \|^4_{H^{\frac{\alpha}{2}}(\mathbb{T})}$$
$$< \infty.$$

Now following the argument in, we can construct a solution on $[0,T]$ for arbitrarily large time T. This is achieved by applying the approximate conservation (11), so as to repeat the local result on successive short time intervals of size δ to reach T, by adjusting the strip width parameter $\sigma \in (0,\sigma_0]$ of the solution according to the size of T. To achieve this first note that by (3.13) and using the fact $E_\sigma(0) \leq E_{\sigma_0}(0)$ which holds for $\sigma \leq \sigma_0$ as e^x is increasing for $x \geq 0$, we obtain

$$(3.14) \quad \sup_{0\leq t\leq T_0} E_\sigma(t) \leq E_\sigma(0) + c\sigma[1+E_\sigma(0)]E^2_{\sigma_0}(0)$$
$$\leq E_{\sigma_0}(0) + c\sigma\left[1+E_{\sigma_0}(0)\right]E^2_{\sigma_0}(0)$$

for some $T_0 \in (0,\delta]$. Thus,

$$(3.15) \qquad \sup_{0\leq t\leq T_0} E_\sigma(t) \leq 2E_{\sigma_0}(0)$$

[2]Since $\alpha/2 \geq 2$ for $\alpha \geq 4$, the Sobolev embedding $H^{\alpha/2} \hookrightarrow L^4$ is well-defined with a constant C_e independent of σ. Specifically, $H^{1/4}(\mathbb{T}) \hookrightarrow L^4(\mathbb{T})$ is the critical embedding, and our regularity $\alpha/2$ exceeds this for all $\alpha \geq 4$.

provided that

$$(21) \qquad c\sigma[1+E_\sigma(0)]E^2_{\sigma_0} \leq E_{\sigma_0}(0).$$

Next, we apply the local theory with initial time $t = T_0$ and time-step size δ to extend the solution from $[0,\tau]$ to $[\tau,\tau+\delta]$. By (3.13) and (3.15), we obtain

$$(3.16) \quad \sup_{\delta\leq t\leq T_0+\delta} E_\sigma(t)$$
$$\leq E_\sigma(T_0) + c\sigma\left[1+2E_{\sigma_0}(0)\right]\left(2E_{\sigma_0}(0)\right)^2.$$

Proceeding this way, we can cover all time intervals $[0,\delta]$, $[\delta,2\delta]$, $[2\delta,3\delta]$, and so on and then apply induction to establish

$$(3.17) \qquad \sup_{0\leq t\leq T} E_\sigma(t) \leq 2E_{\sigma_0}(0) \text{ for } \sigma \geq cT^{-1},$$

where $c > 0$ depends on $E_{\sigma_0}(0)$. This would in turn imply $\sup_{0\leq t\leq T} \| u(t) \|_{H^{\sigma,\frac{\alpha}{2}}(\mathbb{T})} < \infty$ for $\sigma \geq cT^{-1}$.

This completes the proof of Theorem 1.1

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